Two topics around wavelets theory

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Organization of the talk

• Since the Haar works, in the 1910s', we know that there exist orthonormal bases for $L^2(\mathbb{R})$ in the form

$$\{\psi_{j,k}(\bullet),\ (j,k)\in\mathbb{Z}^2\} = \ \big\{2^{\frac{j}{2}}\psi\big(2^j\bullet - k\big),\ (j,k)\in\mathbb{Z}^2\big\}, \tag{1.1}$$

built by dilatation of powers of 2 and by translation by integers of a function ψ named mother function.

• The Haar system is generated by the mother discontinuous function defined for all real x by

$$\psi(x) = \ \mathbb{1}_{(0,\frac{1}{2}]}(x) - \mathbb{1}_{(\frac{1}{2},1]}(x). \tag{1.2}$$

• In the 80s' and the 90s', the construction of smoother bases in this form was systematized by I. Daubechies, S. Mallat and Y. Meyer in the frame of wavelets theory.




Organization of the talk

Definition 2.1

Let $(H, \|\cdot\|)$ a separable Hilbert space. One says that a sequence $\{e_k,\ k \in \mathbb{Z}\}$ forms a Riesz basis of H if it satisfies the following properties:

(1) $\overline{span\{e_k,\ k \in \mathbb{Z}\}} = H$,

(2) $\{e_k,\ k \in \mathbb{Z}\}$ *is a Riesz sequence, that means there are two constants* $0 < c \le c'$ *such that for each complex-valued sequence* $(a_k)_{k\in\mathbb{Z}}$ *with a finite number of non-vanishing terms, one has*

$$c \sum_{k\in\mathbb{Z}} |a_k|^2 \le \Big\| \sum_{k\in\mathbb{Z}} a_k e_k \Big\|^2 \le c' \sum_{k\in\mathbb{Z}} |a_k|^2.$$

Remark 2.1

Any Riesz basis of H is the image of an orthonormal basis of H by an isomorphism of H.

Definition 2.2

A Multi-Resolution Analysis (MRA) of $L^2(\mathbb{R})$ is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying:

(a) *For all* $j \in \mathbb{Z},\ V_j \subset V_{j+1}$,

(b) $\displaystyle\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(c) $\displaystyle\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(\mathbb{R})$,

(d) *For all* $j \in \mathbb{Z},\ f \in V_j \iff f(2 \times \bullet) \in V_{j+1}$,

(e) *There exists* $g \in V_0$ *such that* $\{g(\bullet - k),\ k \in \mathbb{Z}\}$ *is a Riesz basis of* V_0.

Remark 2.2

Observe that there are many sequences of subspaces of $L^2(\mathbb{R})$ which satisfies (a), (b) and (c). However, properties (d) and (e) are specific to the concept of MRA.

Lemma 2.1

Let $h \in L^2(\mathbb{R})$ and $0 < c \leq c'$. The following assertions are equivalent:

(i) $\{h(\bullet - k),\ k \in \mathbb{Z}\}$ *is a Riesz sequence of* $L^2(\mathbb{R})$*, that means for each complex-valued sequence* $(a_k)_{j\in\mathbb{Z}}$ *with finite number of non-vanishing terms, one has*

$$c\sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_{\mathbb{R}} \Big|\sum_{k\in\mathbb{Z}} a_k h(x-k)\Big|^2 dx \leq c'\sum_{k\in\mathbb{Z}} |a_k|^2$$

(ii) *For almost all* $\xi \in \mathbb{R}$*,*

$$c \leq \sum_{k\in\mathbb{Z}} |\widehat{h}(\xi + 2k\pi)|^2 \leq c'.$$

Corollary 2.1

Let $h \in L^2(\mathbb{R})$. The following assertions are equivalent:

(i) $\{h(\bullet - k),\ k \in \mathbb{Z}\}$ *is an orthonormal sequence of* $L^2(\mathbb{R})$*,*

(ii) *For almost all* $\xi \in \mathbb{R}$*,*

$$\sum_{k\in\mathbb{Z}} |\widehat{h}(\xi + 2k\pi)|^2 = 1.$$

Proposition 2.1

(1) *Property (d) in Definition 2.2 means that each space V_j is a dilated version of the reference space V_0,*

$$\text{For all } j \in \mathbb{Z},\ V_j = \left\{ f\left(2^j \times \bullet\right),\ f \in V_0 \right\}.$$

(2) ** Property (e) in Definition 2.2 implies that V_0 is the subspace on $L^2(\mathbb{R})$ of the functions whose Fourier transform can be expressed for almost all $\xi \in \mathbb{R}$ as*

$$\widehat{f}(\xi) = \lambda_f(\xi)\widehat{g}(\xi),$$

where $\lambda_f \in L^2\left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)$.

Proposition 2.2

Let us $(V_j)_{j\in\mathbb{Z}}$ a MRA of $L^2(\mathbb{R})$. For every $j \in \mathbb{Z}$, denote by W_j the subspace of V_{j+1} satisfying

$$V_{j+1} = V_j \overset{\perp}{\oplus} W_j.$$

Then it follows from Proposition 2.1, (1) the fact

$$W_j = \left\{ f\left(2^j \bullet\right),\ f \in W_0 \right\}.$$

Moreover, it follows from property (a),(b) and (c) of Definition 2.2, that for every $J \in \mathbb{Z}$,

$$L^2(\mathbb{R}) = V_J \overset{\perp}{\oplus} \left(\bigoplus_{j=J}^{+\infty} W_j \right), \text{ and } L^2(\mathbb{R}) = \overset{\perp}{\bigoplus_{j\in\mathbb{Z}}} W_j.$$

Example 1

- Assume that for every $j \in \mathbb{Z}$,

$$V_j = \{f \in L^2(\mathbb{R}), \forall k \in \mathbb{Z}, f|_{[\frac{k}{2^j}, \frac{k+1}{2^j})} = \text{constant}\}, \text{ and } g = \mathbb{1}_{[0,1)}.$$

$\{V_j, j \in \mathbb{Z}\}$ forms a MRA on $L^2(\mathbb{R})$ called Haar MRA. In this case, $\{g(\bullet - k),\ k \in \mathbb{Z}\}$ is a Riesz and orthonormal basis of V_0. This Haar MRA is simple, however g being discontinuous, it makes a drawback.

- Regularization of Haar MRA:
  Let $m \geq 1$ be a fixed integer and for every $j \in \mathbb{Z}$, set

$$\mathcal{V}_j = \left\{f \in L^2(\mathbb{R}) \cap \mathcal{C}^{m-1}(\mathbb{R}), \forall k \in \mathbb{Z}, f|_{[\frac{k}{2^j}, \frac{k+1}{2^j})} \in \mathbb{R}_m[x]\right\},$$

, and $g = \mathbb{1}_{[0,1)} \underbrace{* \cdots *}_{m \text{ times}} \mathbb{1}_{[0,1)}$ B-spline of order m.

$\{\mathcal{V}_j, j \in \mathbb{Z}\}$ forms a MRA of $L^2(\mathbb{R})$, and $\text{supp}(g) = [0, m+1)$, and $\{g(\bullet - k),\ k \in \mathbb{Z}\}$ is also a Riesz basis of $\mathcal{V}_0$ but no longer orthonormal.

Starting from a MRA and g, one can always build a function φ such that $\{\varphi(\bullet - k), k \in \mathbb{Z}\}$ forms an orthonormal basis of V_0.

Proposition 2.3

• * *Let* $\varphi \in L^2(\mathbb{R})$ *be the function defined for all* $\xi \in \mathbb{R}$ *by its Fourier transform,*

$$\widehat{\varphi}(\xi) = \frac{\widehat{g}(\xi)}{\Big[\sum_{k \in \mathbb{Z}} |\widehat{g}(\xi + 2\pi k)|^2\Big]^{\frac{1}{2}}}.$$

Then, there exist two sequences $(a_\ell)_\ell, (b_\ell)_\ell \in \ell^2(\mathbb{Z})$ *such that, in* $L^2(\mathbb{R})$,

$$\varphi = \sum_{\ell \in \mathbb{Z}} a_\ell g(\bullet - \ell), \text{ and } g = \sum_{\ell \in \mathbb{Z}} b_\ell \varphi(\bullet - \ell)$$

• *Moreover,* $\{\varphi(\bullet - k), k \in \mathbb{Z}\}$ *is an orthonormal basis of* V_0. *As a consequence, for all* $J \in \mathbb{Z}$, $\{2^{\frac{J}{2}}\varphi(2^J \bullet - k),\ k \in \mathbb{Z}\}$ *is an orthonormal basis of* V_J.

Proposition 2.4

** Let φ be a function such that $\{\varphi(\bullet - k),\ k \in \mathbb{Z}\}$ is an orthonormal basis of V_0. Then there is a function $m_0 \in L^2(\frac{\mathbb{R}}{2\pi\mathbb{Z}})$ satisfying for almost all $\xi \in \mathbb{R}$,*

$$\widehat{\varphi}(2\xi) = m_0(\xi)\widehat{\varphi}(\xi), \tag{2.3}$$

and

$$|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1. \tag{2.4}$$

In particular $m_0(0) = 1$, and $m_0(\pi) = 0$.

φ is usually called <u>scale function</u>, according to (2.3).

m_0 is named <u>quadratic filter.</u>

Hence, for almost $\xi \in \mathbb{R}$,

$$\widehat{\varphi}(\xi) = \prod_{k=1}^{+\infty} m_0\big(2^{-k}\xi\big).$$

Organization of the talk

* Let us fix arbitrarily $\alpha \in \mathbb{R}_+$, and consider

$$\begin{cases} \mathcal{C}^\alpha(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}), \phi \text{ Hölder function of exponent } \alpha\} \\ \mathcal{R}^\alpha(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}), x \longmapsto (1+|x|^\alpha)|\widehat{\phi}(x)| \in L^1(\mathbb{R})\} \\ \mathcal{D}^\alpha(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}), x \longmapsto (1+|x|^\alpha)|\widehat{\phi}(x)| \in L^\infty(\mathbb{R})\} \\ \mathcal{C}_c^\alpha(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}), \phi \text{ Hölder function of exponent } \alpha \text{ with compact support}\} \\ \mathcal{D}_c^\alpha(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}), \text{ with compact support, } x \longmapsto (1+|x|^\alpha)|\widehat{\phi}(x)| \in L^\infty(\mathbb{R})\} \end{cases}.$$

One has

- For all $\varepsilon > 0$, $\mathcal{D}^{\alpha+1+\varepsilon}(\mathbb{R}) \subset \mathcal{R}^\alpha(\mathbb{R}) \subset \mathcal{C}^\alpha(\mathbb{R})$,
- $\mathcal{C}_c^\alpha(\mathbb{R}) \subset \mathcal{D}_c^\alpha(\mathbb{R})$.
- * We eventually make the remark that if $\varphi(\xi) = 2\sum_{k\in\mathbb{Z}} h_k\varphi(2\xi - k)$, with

$(h_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then if the h_k's are null except a finite number, then m_0 is polynomial, and φ has compact support.

* If m_0 is smooth, 2π-periodic, one factorizes on this form

$$m_0(x) = \Big(\frac{1+e^{ix}}{2}\Big)^N p(x),$$

where N is the degree of the root π of m_0.
Hence, for almost all $\xi \in \mathbb{R}$,

$$\big|\widehat{\varphi}(\xi)\big| \; = \left|\frac{2\sin\left(\frac{\xi}{2}\right)}{\xi}\right|^N \pi_1(\xi),$$

where $\pi_1(\xi) := \prod_{k=1}^{+\infty} \big|p\big(2^{-k}\xi\big)\big|$. One sets $u(\xi) := |p(\xi)|$, and for all $n \in \mathbb{N}$,

$$u_n(\xi) := \prod_{k=1}^{n} u\big(2^{-k}\xi\big).$$

Example 2

Choosing p being polynomial allowing to create wavelets bases with compact supports is the Daubechies method.
$(u(\xi))^2 = |p(\xi)|^2$ can be expressed as a polynomial function P with variable $y = \cos^2(\frac{\xi}{2})$, and the relation (2.4) becomes:

$$y^N P(1-y) + (1-y)^N P(y) = 1,$$

with $P \geq 0$ over $[0,1]$. A solution is done by P_N with degree inferior to N:

$$P_N(y) = \sum_{k=0}^{N-1} \binom{N-1+k}{k} y^k.$$

Lemma 3.1

Let's introduce the sequence $(b_n)_{n\in\mathbb{N}}$ *defined for all* $n \in \mathbb{N}$ *by*

$$b_n := \sup_{\xi\in\mathbb{R}} \frac{\log_2 u_n(\xi)}{n}.$$

The sequence $(b_n)_{n\in\mathbb{N}}$ *is sub-additive and so converges to the limit*

$$b := \inf_{n\geq 1} b_n \in \mathbb{R} \cup \{-\infty\}.$$

b is called critic exponent of the quadratic filter m_0.

Theorem 3.1

- *If $\alpha < N - b$, then φ belongs to $\mathcal{D}^{\alpha}(\mathbb{R})$, $\mathcal{R}^{\alpha-1}(\mathbb{R})$, and $\mathcal{C}^{\alpha-1}(\mathbb{R})$.*
- *Assume that $\{\varphi(\bullet - k),\ k \in \mathbb{Z}\}$ is orthonormal and φ compactly supported. then if $\alpha > N - b$ then φ is not Hölder with exponent α.*

Theorem 3.2

If u is continuous, 2π-periodic, Lipschitz on 0, and such that $u(0)=1$, then:

$$b = \inf\{\alpha > 0, \text{there exists } M > 0 \text{ such that for all } \xi \in \mathbb{R},\ \pi_1(\xi) \le M(1+|\xi|^{\alpha})\}$$

By 2π-periodicity, we must study the ergodic product observing for all $\xi \in \mathbb{R}$,

$$\sup_{\xi\in\mathbb{R}} u_n(\xi) = \sup_{\xi\in\mathbb{R}} u_n\left(2^n\xi\right) = \sup_{\xi\in\mathbb{R}} \prod_{k=1}^{n} u_n\left(2^k\xi\right).$$

Precisely, with the transformation $T : \xi \longmapsto 2\xi \mod 2\pi$, studying the behavior of the supremum of an ergodic product.

Now, let X be a metric compact space, and $T : X \longrightarrow X$ a continuous transformation. Let us consider $\mathcal{I}$ the compact convex set of probability measures on X invariant by T.
Let us fix g a real-valued continuous function on X. Let us denote for all $\mu \in \mathcal{I}$,

$$\mu(g) := \int_X g(\xi)\mathrm{d}\mu(\xi).$$

As before, the sequence $\left(\sup_{\xi\in X}\sum_{k=0}^{n-1} g(T^k\xi)\right)_{n\geq 1}$ is sub-additive.
Therefore, the sequence $(\beta_n(g))_{n\geq 1}$ defined for all $n\geq 1$ by

$$\beta_n(g) := \frac{1}{n}\sup_{\xi\in X}\sum_{k=0}^{n-1} g(T^k\xi)$$

converges to a limit denoted $\beta(g)$.

Theorem 3.3

For all $\mu\in\mathcal{I}$, $\beta(g)\geq\mu(g)$.
Moreover, there exists an ergodic measure $\mu_0\in\mathcal{I}$ such that $\beta(g)=\mu_0(g)$
Finally, there exists $x_0\in X$ satisfying

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{n-1} g(T^k x_0)=\beta(g)=\mu_0(g).$$

Now, consider the polynomial function P_N from Daubechies wavelets, as the example before, and its associated trigonometric polynomial function p.

Theorem 3.4

The ergodic probability measure μ_0 maximizing the integral of $\log_2 |p|$ is the measure carried by the periodic points $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Theorem 3.5

Let us denote $\alpha(N)$ the Hölder exponent of the function φ_N associated to the polynomial function P_N. Then we have the estimation:

$$\lim_{N\to+\infty} \frac{\alpha(N)}{N} = 1 - \frac{\log_2(3)}{2}.$$

Finally, let's introduce the Meyer wavelets bases.

Definition 3.1

$\mathcal{S}(\mathbb{R})$, the Schwarz class is the space of infinitely differentiable functions f which satisfy for all $p, N \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} (1+|x|)^N |f^{(p)}(x)| < +\infty.$$

Note that $\mathcal{S}(\mathbb{R})$ is invariant by the Fourier transformation.

We use the same notations as Proposition 2.4. Let $m_1 \in L^2\left(\frac{\mathbb{R}}{2\pi\mathbb{Z}}\right)$ be the function defined for almost all $\xi \in \mathbb{R}$ as

$$m_1(\xi) = e^{-i\xi}\overline{m_0(\xi+\pi)}.$$

Then, one has for almost all $\xi \in \mathbb{R}$,

$$\begin{cases} |m_1(\xi)|^2 + m_1(\xi+\pi)|^2 = 1 \\ m_0(\xi)\overline{m_1(\xi)} + m_0(\xi+\pi)\overline{m_1(\xi+\pi)} = 0 \end{cases}.$$

Theorem 3.6 (Mallat and Meyer, 1985, 1986)

For almost all $\xi \in \mathbb{R}$, *we set*

$$\widehat{\psi}(2\xi) = m_1(\xi)\widehat{\varphi}(\xi).$$

(i) $\{\psi(\bullet - k), k \in \mathbb{Z}\}$ *is an orthonormal basis of* W_0.
(ii) $\{\psi_{j,k},\ j,k \in \mathbb{Z}\} = \ \{2^{\frac{j}{2}}\psi(2^j \bullet - k), j,k \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$.

Theorem 3.7 (Meyer, 1985, 1986)

There exists a scaling function φ *and a mother wavelet* ψ *which generate an orthonormal basis of* $L^2(\mathbb{R})$ *and satisfy the following properties*
(i) $\varphi, \psi \in \mathcal{S}(\mathbb{R})$.

(ii) supp$(\widehat{\varphi}) \subset \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$, *and for all* $\xi \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$, $\widehat{\varphi}(\xi) = 1$.

(iii) supp$(\widehat{\psi}) \subset \left[-\frac{8\pi}{3}, -\frac{2\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{8\pi}{3}\right]$. *(*$\widehat{\psi}$ *is null on a neighborhood of* 0.*)*
A wavelet basis generated by such φ *and* ψ *is called a Meyer wavelet basis.*
Moreover, $\{\widehat{\psi}_{j,k}, j,k \in \mathbb{Z}\}$, *is also an orthonormal basis of* $L^2(\mathbb{R})$.

Organization of the talk

Let's begin with an historical example, using Haar basis, and Brownian Motion (BM). Now let $\{W(t) : t \in [0,1]\}$ be a BM over $[0,1]$. It can be expressed as the Wiener integral,

$$W(t) = \int_0^1 \mathbb{1}_{[0,t]}(s) \mathrm{d}W(s)$$

By expanding the function $\mathbb{1}_{[0;t]}(s)$ in the Haar system, it follows that, in $L^2(\mathbb{R})$,

$$\mathbb{1}_{[0,t]}(s) = t\mathbb{1}_{[0,1]}(s) + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} 2^{-\frac{j}{2}} \tau\big(2^j t - k\big) 2^{\frac{j}{2}} \psi\big(2^j s - k\big),$$

where τ is the triangle function based on $[0,1]$ such that $\tau(\frac{1}{2}) = \frac{1}{2}$.
By isometry property of Wiener measure, one gets, with a.s uniform convergence,

$$W(t) = t\varepsilon_0 + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} 2^{-\frac{j}{2}} \tau\big(2^j t - k\big) \varepsilon_{j,k},$$

where $\varepsilon_0 = W(1) \sim \mathcal{N}(0,1)$, and $\varepsilon_{j,k} = 2^{\frac{j}{2}} \int_0^1 \psi\big(2^j s - k\big) \mathrm{d}W(s) \sim \mathcal{N}(0,1)$ if $j > 0$ and $0 \leq k \leq 2^j - 1$.

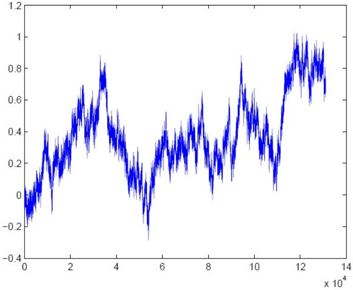
$(\Omega, \mathcal{A}, \mathbb{P})$ is probability complete space. Let us now introduce Multifractional Brownian Motion (MBM) $Z = \{Z(t),\ t \in \mathbb{R}\}$, by its harmonizable integral. Consider $H : \mathbb{R} \longrightarrow (0,1)$ a function such that $H(\mathbb{R}) \subset [\underline{H}, \overline{H}] \subset (0,1)$, then one defines for all $t \in \mathbb{R}$

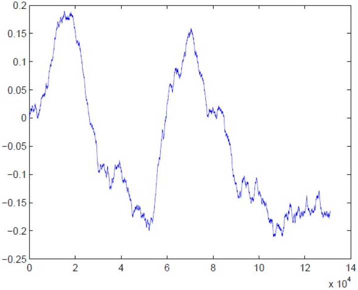
$$Z(t) =^* \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{(i\xi)^{H(t)+\frac{1}{2}}} \mathrm{d}\widehat{W}(\xi),$$

(Jaffard, Peltier and Roux harmonizable representation, 90s'), where $\widehat{W}$ is the orthogonally scattered Wiener measure (it can be considered as the Fourier transform of the Wiener measure).

• Modulo a constant, if H is a constant function in $(0,1)$, then Z is the Fractional Brownian Motion (FBM) with Hurst index H. And if $H = \frac{1}{2}$, we find the Brownian Motion.
• FBM was introduced in 1940 by Kolmogorov and made popular in 60s' by Mandelbrot and Van Ness. It has turned out to be a powerful tool in modeling and has been applied in many areas (Finance, Hydrology, Signals et Images processing, Telecommunication, among other domains).

• Despite its usefulness, FBM model has some limitations, an important one of them is that the roughness of its path remains everywhere the same: the pointwise Hölder exponent of the FBM Z is almost surely equal to H everywhere.

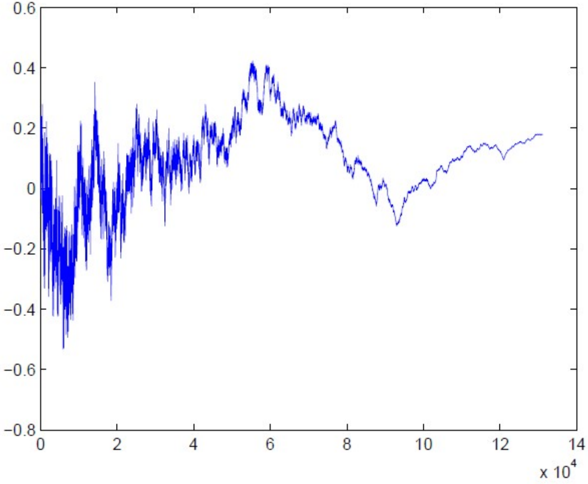




Here are two simulations of FBM, on the left with $H = 0.3$ on the right with $H = 0.7$.

• Making variable H is an idea in the 90s'.

• Simulation of MBM with $H(t) = 0.1 + 0.06t$ (remark that there exists simulation methods using wavelets for FBM and MBM).



Let the field $X = \{X(u,v), (u,v) \in \mathbb{R} \times (0,1)\}$ defined for all $(u,v) \in \mathbb{R} \times (0,1)$ by

$$X(u,v) := \int_{\mathbb{R}} \frac{e^{iu\cdot\xi} - 1}{(i\xi)^{v+\frac{1}{2}}} \mathrm{d}\widehat{W}(\xi). \tag{4.5}$$

X is called *the field generating the MBM Z* since

$$\forall t \in \mathbb{R}, Z(t) = X(t, H(t)). \tag{4.6}$$

Considering (4.6), properties of Z are strongly influenced by those of X.

• For all $(j,k)\in\mathbb{Z}^2$, one sets $\varepsilon_{j,k} := \int_{\mathbb{R}} \overline{\widehat{\psi}_{j,k}(\xi)}\,\mathrm{d}\widehat{W}(\xi)$.

By the definitions of $\widehat{W}$ and $\widehat{\psi}_{j,k}$, since

$$\mathrm{supp}(\widehat{\psi}_{j,k}) \subset \Big[-\frac{2^{j+3}\pi}{3}, \frac{2^{j+3}\pi}{3}\Big] \setminus \Big(-\frac{2^{j+1}\pi}{3}, \frac{2^{j+1}\pi}{3}\Big),$$

the sequence $(\varepsilon_{j,k})_{(j,k)\in\mathbb{Z}^2}$ is i.i.d $\mathcal{N}(0,1)$.

• One denotes Ψ the deterministic function defined for all $(x,v)\in\mathbb{R}^2$ by

$$\Psi(x,v) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\eta}\frac{\widehat{\psi}(\eta)}{|\eta|^{v+\frac{1}{2}}}\mathrm{d}\eta.$$

We have the "well-localization" in y uniformly in v restricted to any compact interval of $\mathbb{R}$ for this function. More precisely, for all $n,m\in\mathbb{Z}_+$, for all $L\in\mathbb{N}$, for all $M>0$, one has

$$\sup\Big\{(3+|x|)^L\big|\partial_x^n\partial_v^m\Psi(x,v)\big|,\ (x,v)\in\mathbb{R}\times[-M,M]\Big\} < +\infty.$$

Theorem 4.1 (wavelet representation of Generator of MBM)

The generator of MBF $X = \{X(u,v),\ (u,v) \in \mathbb{R} \times (0,1)\}$ *can be represented as*

$$X(u,v) = \sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv}\big[\Psi\big(2^j u - k, v\big) - \Psi\big(-k, v\big)\big]\varepsilon_{j,k},$$

where the series is, on a event $\widetilde{\Omega}$ *of probability* 1, *uniformly convergent in* (u,v), *on each compact interval of* $\mathbb{R} \times (0,1)$.

Corollary 4.1

One sets, on $\widetilde{\Omega}$,

$$\widetilde{Z}(t) = \sum_{(j,k)\in\mathbb{Z}^2} 2^{-jv}\big[\Psi\big(2^j t - k, H(t)\big) - \Psi\big(-k, H(t)\big)\big]\varepsilon_{j,k},$$

$\widetilde{Z}$ *is a continuous version of* $Z \iff H$ *is continuous.*

With this modification and its properties, one obtains

Theorem 4.2

If $H \in \mathcal{C}^{\gamma}(\mathbb{R})$, with $\gamma \in (\overline{H}, 1)$ then

$$\mathbb{P}\big(\forall \tau \in \mathbb{R}, \varrho_{\widetilde{Z}}(\tau) = H(t)\big) = 1,$$

where the random variable $\varrho_{\widetilde{Z}}(\tau)$ denotes the pointwise Hölder exponent on τ of $\widetilde{Z}$.

The choice of the basis $\widehat{\psi}_{j,k}$ to build $\widetilde{Z}$ is crucial to get this statement.

References:

• Ayache, A., Multifractional Stochastic Fields: Wavelet Strategies In Multifractional Frameworks, World Scientific (2018)

• Ayache, A., A mini-course on Wavelets and Fractional Process at National Taiwan University (December 2009) and at the Faculty of Sciences of Monastir in Tunisia (January 2011), https://math.univ-lille1.fr/ ayache/

• Cohen, A., Conze, J.P., Régularité des bases d'ondelettes et mesures ergodiques, Revista Matematica Iberoamericana Vol. 8, n°3, 1992

• Daubechies, I., Orthonormal basis of compactly supported wavelets. Comm. Pure Appl. Math. 41 (1988), 909-996.

• Mallat, S., Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$. Trans. Math. Soc. 315 (1989), 69-88.

• Meyer, Y., Wavelets and operators. Cambridge University Press, Cambridge, UK (1992)