Harmonizable Multifractional Stable Field: Wavelet representation and sample path behavior

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Organization of the talk

$(\Omega, \mathcal{G}, \mathbb{P})$ is a complete probability space.

The integer N is arbitrary and fixed, and $|\cdot|_2$ denotes the Euclidean norm on $\mathbb{R}^N$.

Let H be a function on $\mathbb{R}^N$ with values in an arbitrary compact interval $[\underline{H}, \overline{H}] \subset (0,1)$.

Definition 1 (Multifractional Brownian Field (MBF))

For all $t \in \mathbb{R}^N$,

$$Z_B(t) = \int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|_2^{H(t)+N/2}} \mathrm{d}\widehat{W}(\xi), \tag{1.1}$$

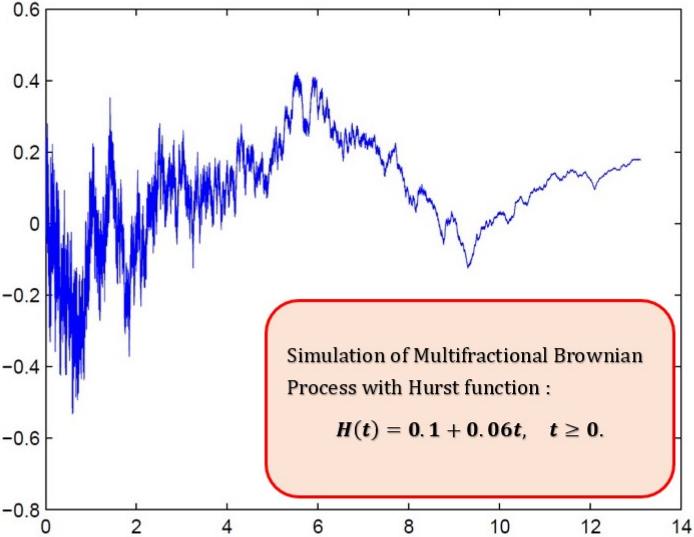
where $\widehat{W}$ is the orthogonally scattered measure of a Brownian measure W with Lebesgue control measure on $\mathbb{R}^N$.

• When $N = 1$ and H is a constant function (Hurst parameter), Z_B is the Fractional Brownian Process (FBP) introduced by Kolmogorov in the 40s', made famous by Mandelbrot and Van-Ness in the 70s'. It was shown that the pointwise Hölder exponent of FBF is almost surely equal to H on every point on $\mathbb{R}^N$. And when $H = \frac{1}{2}$, we recognize the Brownian motion.

• Making variable H was an idea introduced in the 90s' (Benassi, Jaffard, Peltier, Levy-Vehel, Roux). The following simulation illustrates that regularity of Z_B is controlled by the Hurst function.

Precisely, if we assume for any fixed $\tau \in \mathbb{R}^N$, $\lim\limits_{\mathbf{t} \underset{\neq}{\longrightarrow} \tau} \dfrac{H(t) - H(\tau)}{|t - \tau|_2} = 0$, then the pointwise Hölder exponent on τ of MBF is almost surely equal to $H(\tau)$.
(See e.g., Ayache (2018), Theorem 6.17.)

Simulation for a Multifractional Brownian Process with a choice of H



Organization of the talk

Let $\alpha \in (0,2)$ be the stability parameter.

Definition 2 (The Harmonizable Multifractional Stable Field (HMSF))

For all $t \in \mathbb{R}^N$,

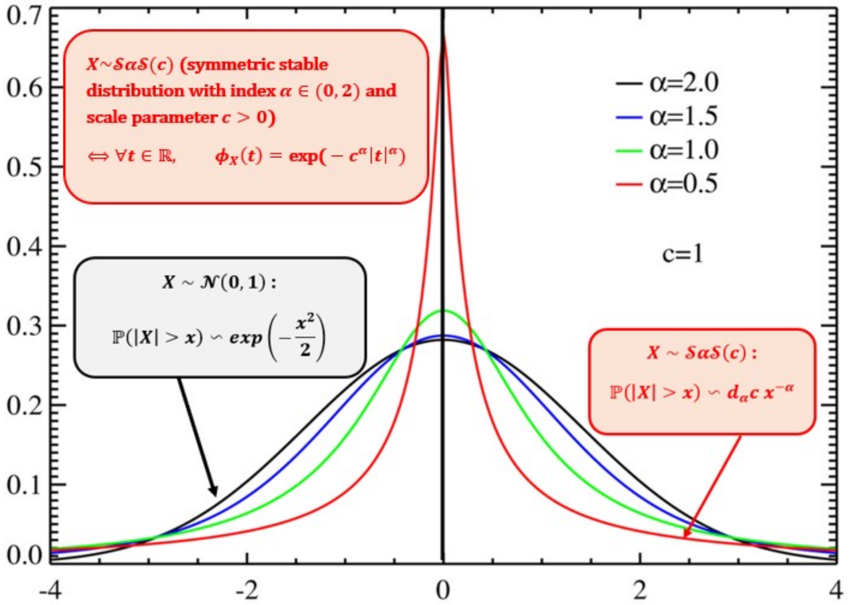
$$Z(t) := \mathfrak{Re}\Big[\int_{\mathbb{R}^N} \frac{e^{it\cdot\xi} - 1}{|\xi|_2^{H(t)+\frac{N}{\alpha}}} \mathrm{d}\widetilde{\mathcal{M}}_\alpha(\xi) \Big], \tag{2.2}$$

where $\widetilde{\mathcal{M}}_\alpha$ is the complex-valued isotropic SαS random measure on $\mathbb{R}^N$ with Lebesgue control measure, and where $t\cdot\xi$ is the inner product between t and ξ on $\mathbb{R}^N$.

A general question for such a non-Gaussian extension is to know if one preserves the regularity property as MBF. In this talk, we will bring an answer.

According to the following frame, the marginal symmetric α-stable distributions for such extension, are heavy-tailed, and only have moments of order $0 \le \gamma < \alpha$.

(The Gaussian aspect of the marginal distributions of MBF was crucial to obtain the regularity of its sample paths.)


$X \sim \mathcal{S}\alpha\mathcal{S}(c)$ (symmetric stable distribution with index $\alpha \in (0,2)$ and scale parameter $c > 0$)
$\Leftrightarrow \forall t \in \mathbb{R}, \quad \phi_X(t) = \exp(-c^\alpha |t|^\alpha)$
α=2.0
α=1.5
α=1.0
α=0.5
c=1
$X \sim \mathcal{N}(0,1)$:
$\mathbb{P}(|X| > x) \backsim exp\left(-\frac{x^2}{2}\right)$
$X \sim \mathcal{S}\alpha\mathcal{S}(c)$:
$\mathbb{P}(|X| > x) \backsim d_\alpha c\, x^{-\alpha}$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-4
-2
0
2
4

We mention by passing that there is another stable extension of MBF called the Linear Multifractional Stable Field (LMSF) defined for all $t \in \mathbb{R}^N$ as

$$Y(t) := \int_{\mathbb{R}^N} \Big(|t-s|_2^{H(t)-\frac{N}{\alpha}} - |s|_2^{H(t)-\frac{N}{\alpha}} \Big) \mathrm{d}M_\alpha(s), \tag{2.3}$$

where M_α is any real-valued SαS random measure with Lebesgue control measure on $\mathbb{R}^N$.

If $N = 1$, and for some $a < b$ if $\sup_{t\in(a,b)} H(t) < \frac{1}{\alpha}$, then it can be shown (see Stoev and Taqqu (2004)) that every version of Y has unbounded paths on any sub-interval $(a', b') \subset (a, b)$ of positive length.

We believe that this theorem can be extended to $N \geq 2$.

This is why we are going to focus on HMSF whose, sample paths are continuous functions as soon as H is continuous, as the MBF does. (See e.g., Ayache (2018).)

Organization of the talk

Dozzi and Shevchenko (2011) introduced the Harmonizable Multifractional Stable Process (HMSP) in the case $N = 1$ and $\alpha \in (1, 2)$.

Thanks to a LePage series representation of the process Z (see e.g., Samorodnitsky and Taqqu (1994)), they obtained what follows.

Assume that H is a Hölder function of order $\gamma > \overline{H}$. Then Z has a version whose is almost surely Hölder continuous of any order $\gamma < \underline{H}$, and moreover almost surely, for all $T, \eta > 0$, satisfies

$$\sup_{\begin{cases}(t,s)\in[0,T]) \\ |t-s|<\delta\end{cases}} |Z(t) - Z(s)| = o\Big(\delta^{\underline{H}} |\log(\delta)|^{\frac{1}{\alpha}+\frac{1}{2}+\eta}\Big),\ \delta \longrightarrow 0^+. \tag{3.4}$$

One of the goal of our talk is to show that the power $\frac{1}{\alpha} + \frac{1}{2} + \eta$ of the log is not optimal and can be substituted by $\frac{1}{\alpha} + \eta$.

Biermé, Lacaux and Scheffler (2011) introduced a large class of harmonizable multi-operator scaling stable random fields including the HMSF Z with $\alpha \in (0,2)$ and with any $N \geq 1$.
They obtained some results for this class which implies what follows.

Assuming H is a locally Lipschitz function, one has :
(i) On any non-empty compact interval I of $\mathbb{R}^N$, sample paths of Z are almost surely Hölder functions of any order $\gamma < \underline{H}(I) := \min_{t \in I} H(t)$,
This last result is also obtained by using LePage series expansion for Z.

(ii) One has

$$\forall \tau \in \mathbb{R}^N, \mathbb{P}\big(\rho_Z(\tau) = H(\tau)\big) = 1, \tag{3.5}$$

where

$$\rho_Z(\tau) := \sup\left\{\gamma \in [0,1], \limsup_{t \longrightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t-\tau|_2^{\gamma}} < +\infty\right\} \tag{3.6}$$

denotes the pointwise Hölder exponent on τ of Z.

The three main goals of our talk are the following.

Goal 1

To obtain, under weaker assumptions on H than locally Lipschitz-continuity, optimal uniform and pointwise moduli of continuity for Z.

Goal 2

To show, under a weaker assumption on H than locally Lipschitz-continuity, that

$$\mathbb{P}\big(\forall \tau \in \mathbb{R}^N, \rho_Z(\tau) = H(\tau)\big) = 1, \tag{3.7}$$

which is significantly better than :

$$\forall \tau \in \mathbb{R}^N : \mathbb{P}\big(\rho_Z(\tau) = H(\tau)\big) = 1. \tag{3.8}$$

Goal 3

Finally, to derive an almost sure estimate for the asymptotic behavior of Z at infinity, and to find some assumptions on H for having optimality for this estimate.

Organization of the talk

Let the field $X = \{X(u,v), (u,v) \in \mathbb{R}^N \times (0,1)\}$ defined for all $(u,v) \in \mathbb{R}^N \times (0,1)$ by

$$X(u,v) := \mathfrak{Re}\Big[\int_{\mathbb{R}^N} F_\alpha(u,v,\xi) \mathrm{d}\widetilde{\mathcal{M}}_\alpha(\xi) \Big], \tag{4.9}$$

where F_α is the kernel function defined for all $(u,v) \in \mathbb{R}^N \times (0,1)$ and $\xi \in \mathbb{R}^N \setminus \{0\}$ by

$$F_\alpha(u,v,\xi) := \frac{e^{iu\cdot\xi} - 1}{|\xi|_2^{v+\frac{N}{\alpha}}} \text{ and } F_\alpha(u,v,0) = 0. \tag{4.10}$$

X is called *the field generating the HMSF Z* since

$$\forall t \in \mathbb{R}^N, Z(t) = X(t, H(t)). \tag{4.11}$$

Considering (4.11), properties of Z are strongly influenced by those of X.

Let $\Upsilon_* = \{1, \cdots, 2^N - 1\}$, and let the sequence $\left(\psi_{\delta,j,k}\right)_{(\delta,j,k)\in\Upsilon_*\times\mathbb{Z}\times\mathbb{Z}^N}$ be a Meyer orthonormal wavelet basis for $L^2(\mathbb{R}^N)$. Notice that

$$\psi_{\delta,j,k}(x) = 2^{j\frac{N}{2}}\psi_\delta\left(2^j x - k\right), \tag{4.12}$$

where ψ_δ, $\delta \in \Upsilon_*$ are the mother wavelets.

The sequence $\left(\overline{\widehat{\psi}_{\delta,j,k}}\right)_{(\delta,j,k)\in\Upsilon_*\times\mathbb{Z}\times\mathbb{Z}^N}$ of the complex conjugates of the Fourier transforms of the $\psi_{\delta,j,k}$ is also an orthonormal basis for $L^2(\mathbb{R}^N)$, but it is not a basis for $L^\alpha(\mathbb{R}^N)$ if $\alpha \in (0, 2)$.

In spit of the fact that the kernel function $F_\alpha(u,v,\cdot)$ associated to X doesn't belong to $L^2(\mathbb{R}^N)$, we manage to show that it can be decomposed in $L^\alpha(\mathbb{R}^N)$ on the sequence $\left(\overline{\widehat{\psi}_{\delta,j,k}}\right)_{(\delta,j,k)\in\Upsilon_*\times\mathbb{Z}\times\mathbb{Z}^N}$. Which allows obtaining :

Theorem 3

There exists an event Ω_α^* of probability 1 such that for all $(u,v,\omega)\in\mathbb{R}^N\times(0,1)\times\Omega_\alpha^*$, one has, with absolute convergence :

$$X(u,v,\omega)=\sum_{(\delta,j,k)\in\Upsilon_*\times\mathbb{Z}\times\mathbb{Z}^N} 2^{-jv}\left[\Psi_\delta^{(\alpha)}(2^j u-k,v)-\Psi_\delta^{(\alpha)}(-k,v)\right]\varepsilon_{\delta,j,k}^{(\alpha)}(\omega). \quad (4.13)$$

$\Psi_\delta^{(\alpha)}$ are real-valued deterministic C^∞ functions on $\mathbb{R}^N\times\mathbb{R}$ such that $\Psi_\delta^{(\alpha)}(\bullet,v)\in\mathcal{S}(\mathbb{R}^N)$, for all $v\in\mathbb{R}$.

$\|\cdot\|_\alpha$ being the usual (quasi-)norm on $L^\alpha(\mathbb{R}^N)$, one has

$$\varepsilon_{\delta,j,k}^{(\alpha)}=\mathfrak{Re}\left[\int_{\mathbb{R}^N}\overline{\widehat{\psi}_{\delta,j,k}(\xi)}\,d\widetilde{\mathcal{M}}_\alpha(\xi)\right]\sim S\alpha S\left(\|\psi_\delta\|_\alpha\right). \quad (4.14)$$

The following crucial lemma, which provides estimates for the random variables $\varepsilon^{(\alpha)}_{\delta,j,k}$, is inspired by some results in (Ayache and Boutard 2017) and (Ayache and Xiao 2024).

Similarly to them, it is obtained by using LePage series representation for the real-valued SαS stochastic field $\left(\varepsilon^{(\alpha)}_{\delta,j,k}\right)_{(\delta,j,k)\in\Upsilon_*\times\mathbb{Z}\times\mathbb{Z}^N}$.

Lemma 4

For each $\alpha \in (0,2)$, for all $\eta > 0$, there exists a positive random variable C such that for all $(\delta, j, k) \in \Upsilon_* \times \mathbb{Z} \times \mathbb{Z}^N$, on the event Ω_α^* of probability 1, one has

$$\left|\varepsilon_{\delta,j,k}^{(\alpha)}\right| \le C(1+|j|)^{\frac{1}{\alpha}+\eta} \log^{\frac{\lfloor \alpha \rfloor}{2}}(3+|j|+|k|_1), \tag{4.15}$$

where $\lfloor \alpha \rfloor$ denotes the integer part of α, $|k|_1 := |k_1| + \cdots + |k_N|$.

Moreover, when $\alpha \in [1,2)$, $\vartheta > 0$ is an arbitrary constant and one restricts to $j \in \mathbb{Z}_+$ and $k \in \mathbb{Z}^N$ satisfying

$$|k|_\infty := \max_{1 \le r \le N} |k_r| \le \vartheta 2^j, \tag{4.16}$$

then the following significantly improved version of the inequality (4.15) holds on Ω_α^* for all $\eta > 0$ and $\delta \in \Upsilon_*$:

$$\left|\varepsilon_{\delta,j,k}^{(\alpha)}\right| \le C(1+j)^{\frac{1}{\alpha}+\eta}. \tag{4.17}$$

Thanks to Lemma 4, it turns out that the random wavelet series representing the field X converges in a much stronger way than the one we have already seen.

Theorem 5

When their partial sums are well-chosen, the random wavelet series representing the field X, and all its term by term any order partial derivative with respect to v, are, on the event Ω_α^* of probability 1, uniformly convergent over all compact boxes on $\mathbb{R}^N \times (0,1)$.

Therefore, X has a version whose sample paths are almost surely continuous functions on $\mathbb{R}^N \times (0,1)$ and $\mathcal{C}^\infty$ with respect to $v \in (0,1)$.

Corollary 6

A sufficient condition for the HMSF Z to have continuous sample paths on $\mathbb{R}^N$ is that the Hurst functional H be continuous on $\mathbb{R}^N$.

Moreover, when H is discontinuous at some point $\tau \in \mathbb{R}^N \setminus \{0\}$, then, with probability 1, sample paths of Z are discontinuous functions at τ.

Organization of the talk

The Hurst function H is assumed to be continuous on $\mathbb{R}^N$.

Theorem 7 (Global modulus of continuity)

Let I be an arbitrary non-empty fixed compact box of $\mathbb{R}^N$.
One sets $\underline{H}(I) := \min_{t\in I} H(t)$.

Moreover, assume the continuous Hurst function H satisfies the following **uniform Hölder condition :** *for some finite c, for all* $(t^{(1)}, t^{(2)}) \in I^2$,

$$\left|H(t^{(1)}) - H(t^{(2)})\right| \leq c\left|t^{(1)} - t^{(2)}\right|_1^{\underline{H}(I)} \log^{\frac{1}{\alpha}}\left(1 + \left|t^{(1)} - t^{(2)}\right|_1^{-1}\right). \tag{5.18}$$

Then, on the event Ω_α^* of probability 1*, for all $\eta > 0$, one has*

$$\sup_{(t^{(1)}),t^{(2)})\in I^2} \frac{\left|Z\left(t^{(1)}\right) - Z\left(t^{(2)}\right)\right|}{\left|t^{(1)} - t^{(2)}\right|_1^{\underline{H}(I)} \log^{\frac{1}{\alpha}+\eta}\left(1 + \left|t^{(1)} - t^{(2)}\right|_1^{-1}\right)} < +\infty. \tag{5.19}$$

Sketch of the proof of Theorem 7

First step : one shows the following statement for the generating field X.

Theorem 8 (Global modulus of continuity for X)

Let $\varrho > 0$ and $0 < a < b < 1$ be arbitrary and fixed. Then, one has on the event Ω_α^* of probability 1, for all $\eta > 0$,

$$\sup_{\substack{(u^{(1)},u^{(2)})\in[-\varrho,\varrho]^N \\ (v_1,v_2)\in[a,b]}} \frac{\left|X(u^{(1)},v_1)-X(u^{(2)},v_2)\right|}{\left|u^{(1)}-u^{(2)}\right|_1^{v_1\vee v_2}\log^{\frac{1}{\alpha}+\eta}\left(1+\left|u^{(1)}-u^{(2)}\right|_1^{-1}\right)+|v_1-v_2|} < +\infty, \qquad (5.20)$$

where $v_1 \vee v_2 := \sup\{v_1, v_2\}$.

To prove it, one writes for all $(u, v, \omega) \in Q_{\varrho,a,b} \times \Omega^*_\alpha$,

$$X(u, v, \omega) = X^-(u, v, \omega) + X^+(u, v, \omega)$$

where $X^+(u, v, \omega)$ is called high frequency and denotes

$$X^+(u, v, \omega) := \sum_{(\delta,j,k)\in\Upsilon_*\times\mathbb{Z}_+\times\mathbb{Z}^N} 2^{-jv}\left[\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)\right]\varepsilon_{\delta,j,k}^{(\alpha)}(\omega),$$

and $X^-(u, v, \omega)$ is the low frequency (the same sum as $X^+(u, v, \omega)$ but indexed on $j \in (-\mathbb{N})$).

• Among properties showed to obtain a version of X whose sample paths are almost surely continuous function on $\mathbb{R}^N \times (0, 1)$, $\mathcal{C}^\infty$ with respect to $v \in (0, 1)$ (Theorem 5), one showed that X^- is $\mathcal{C}^\infty$ on $\mathbb{R}^N \times (0, 1)$. So (5.20) is satisfied if X is substituted by X^-.

• Moreover, one showed X^+ is $\mathcal{C}^\infty$ with respect to $v \in (0,1)$. Let $\left(u^{(1)}, v_1\right), \left(u^{(2)}, v_2\right) \in Q_{\varrho,a,b}$ be arbitrary, there is no restriction to assume that $0 < \left|u^{(1)} - u^{(2)}\right| \le 1$ and $v_1 \vee v_2 = v_1$. One gets, on the event Ω^*_α, that

$$\begin{aligned}
&\left|X^+\left(u^{(1)}, v_1\right) - X^+\left(u^{(2)}, v_2\right)\right| \\
&\le \left|X^+\left(u^{(1)}, v_1\right) - X^+\left(u^{(2)}, v_1\right)\right| + \left|X^+\left(u^{(2)}, v_1\right) - X^+\left(u^{(2)}, v_2\right)\right| \\
&\le \left|X^+\left(u^{(1)}, v_1\right) - X^+\left(u^{(2)}, v_1\right)\right| + C_1|v_1 - v_2|,
\end{aligned} \tag{5.21}$$

where C_1 is a positive finite random variable not depending on $\left(u^{(1)}, v_1\right)$ and $\left(u^{(2)}, v_2\right)$. In view of (5.21) and of the fact that $v_1 \vee v_2 = v_1$, it is enough to show that, for some positive finite random variable C_2, not depending on $\left(u^{(1)}, v_1\right)$ and $\left(u^{(2)}, v_2\right)$, one has on Ω^*_α,

$$\left|X^+\left(u^{(1)}, v_1\right) - X^+\left(u^{(2)}, v_1\right)\right| \le C_2\left|u^{(1)} - u^{(2)}\right|^{v_1} \log^{\frac{1}{\alpha}+\eta}\left(1 + \left|u^{(1)} - u^{(2)}\right|_1^{-1}\right). \tag{5.22}$$

- Next, consider for all $j \in \mathbb{Z}$, $(u, v, \omega) \in (0,1) \times [-\varrho, \varrho]^N \times \Omega_\alpha^*$,

$$X_j(u, v, \omega) := 2^{-jv} \sum_{(\delta,k)\in\Upsilon_*\times\mathbb{Z}^N} \Big[\Psi_\delta^{(\alpha)}(2^j u - k, v) - \Psi_\delta^{(\alpha)}(-k, v)\Big] \varepsilon_{\delta,j,k}^{(\alpha)}(\omega). \quad (5.23)$$

One proved one more time in view of Theorem 5, the two following inequalities. For all $j \in \mathbb{Z}$, $u^{(1)}, u^{(2)} \in [-\varrho, \varrho]^N$, $v \in [a, b]$, and $\omega \in \Omega_\alpha^*$, for any fixed $\eta > 0$, there exists $C_3(\omega), C_5(\omega) > 0$ (not depending on j, v and $u^{(1)}$, $u^{(2)}$)

$$\Big|X_j\big(u^{(1)}, v, \omega\big) - X_j\big(u^{(2)}, v, \omega\big)\Big| \le C_3(\omega) 2^{j(1-v)} (1 + |j|)^{\frac{1}{\alpha}+\eta} \Big|u^{(1)} - u^{(2)}\Big|_1. \quad (5.24)$$

and,

$$\Big|X_j\big(u^{(1)}, v, \omega\big)\Big| \le 2^{-jv} C_5(\omega) (1 + |j|)^{\frac{1}{\alpha}+\eta}. \quad (5.25)$$

- Since $0 < \left|u^{(1)} - u^{(2)}\right| \leq 1$ there is a unique $j_0 \in \mathbb{Z}_+$ satisfying

$$2^{-(j_0+1)} < \left|u^{(1)} - u^{(2)}\right|_1 \leq 2^{-j_0}. \tag{5.26}$$

In other words, j_0 is the unique non-negative integer such that

$$j_0 \leq \frac{\log\left(\left|u^{(1)} - u^{(2)}\right|_1^{-1}\right)}{\log(2)} < j_0 + 1. \tag{5.27}$$

- Next, notice that, using the triangle inequality, one has that

$$\left|X^+\left(u^{(1)}, v_1\right) - X^+\left(u^{(2)}, v_1\right)\right| \leq R_{j_0}\left(u^{(1)}, u^{(2)}, v_1\right) + S_{j_0}\left(u^{(1)}, u^{(2)}, v_1\right), \tag{5.28}$$

where

$$R_{j_0}\left(u^{(1)}, u^{(2)}, v_1\right) := \sum_{j=0}^{j_0} \left|X_j\left(u^{(1)}, v_1\right) - X_j\left(u^{(2)}, v_1\right)\right|, \tag{5.29}$$

and

$$S_{j_0}\left(u^{(1)}, u^{(2)}, v_1\right) := \sum_{j=j_0+1}^{+\infty} \left|X_j\left(u^{(1)}, v_1\right) - X_j\left(u^{(2)}, v_1\right)\right|. \tag{5.30}$$

• Finally, combining the characterizations (5.26) and (5.27) of j_0, (5.24) about every X_j and the fact $v_1 \leq b$, on the event Ω_α^*, on one hand, one has that

$$\begin{aligned} R_{j_0}\big(u^{(1)}, u^{(2)}, v_1\big) &\leq C_3 \big|u^{(1)} - u^{(2)}\big|_1 \sum_{j=0}^{j_0} 2^{j(1-v_1)}(1+j)^{\frac{1}{\alpha}+\eta} \\ &\leq C_4 \big|u^{(1)} - u^{(2)}\big|_1^{v_1} \log^{\frac{1}{\alpha}+\eta}\Big(1 + \big|u^{(1)} - u^{(2)}\big|_1^{-1}\Big), \end{aligned} \tag{5.31}$$

and on the other hand, using this time (5.25),

$$\begin{aligned} S_{j_0}\big(u^{(1)}, u^{(2)}, v_1\big) &\leq C_5 \sum_{j=j_0+1}^{+\infty} 2^{-jv_1}(1+j)^{\frac{1}{\alpha}+\eta} \\ &\leq C_6 \big|u^{(1)} - u^{(2)}\big|_1^{v_1} \log^{\frac{1}{\alpha}+\eta}\Big(1 + \big|u^{(1)} - u^{(2)}\big|_1^{-1}\Big). \end{aligned} \tag{5.32}$$

The two last inequalities permit us to conclude the proof of Theorem 8.

Second step Using the equality $Z(t) = X(t, H(t))$ and Theorem 8 with $a = \underline{H}$, $b = \overline{H}$, a fixed $\varrho \geq 1$ such that $I \subset [-\varrho, \varrho]^N$, and any fixed $\eta > 0$, it follows that, for some positive finite random variable C_1 one has, on the event Ω_α^*, for all $\big(t^{(1)}, t^{(2)}\big) \in I^2$,

$$\big|Z((t^{(1)}) - Z(t^{(2)})\big| \tag{5.33}$$
$$\leq C_1\Big(\big|t^{(1)} - t^{(2)}\big|_1^{H(t^{(1)})\vee H(t^{(2)})} \log^{\frac{1}{\alpha}+\eta}\big(1 + |t^{(1)} - t^{(2)}|_1^{-1}\big) + \big|H\big(t^{(1)}\big) - H\big(t^{(2)}\big)\big|\Big).$$

Moreover, since $(2N\varrho)^{-1}\big|t^{(1)} - t^{(2)}\big|_1 \leq 1$, $\underline{H}(I)$ being the minimum of H on I, one can derive that

$$\big|t^{(1)} - t^{(2)}\big|_1^{H(t^{(1)})\vee H(t^{(2)})} = (2N\varrho)^{H(t^{(1)})\vee H(t^{(2)})}\Big((2N\varrho)^{-1}\big|t^{(1)} - t^{(2)}\big|_1\Big)^{H(t^{(1)})\vee H(t^{(2)})}$$
$$\leq (2N\varrho)^{\overline{H}-\underline{H}}\,\big|t^{(1)} - t^{(2)}\big|_1^{\underline{H}(I)}. \tag{5.34}$$

Finally combining (5.33) and (5.34), with the uniform Hölder condition of H on I for the term $\big|H\big(t^{(1)}\big) - H\big(t^{(2)}\big)\big|$, one obtains the conclusion of Theorem 7. □

Theorem 9 (Optimality for global modulus)

Let I be an arbitrary non-empty compact box of $\mathbb{R}^N$.
We assume that H satisfies on I the same **uniform Hölder condition** *(5.18) as before.*

Moreover, assume that there exists some point $\tau^{(0)} \in \mathring{I}$ (the interior of I) such that

$$H\big(\tau^{(0)}\big) = \underline{H}(I) := \min_{t \in I} H(t). \tag{5.35}$$

Then, almost surely, one has

$$\sup_{(t^{(1)},t^{(2)}) \in I^2} \frac{\left|Z\big(t^{(1)}\big) - Z\big(t^{(2)}\big)\right|}{\left|t^{(1)} - t^{(2)}\right|_1^{\underline{H}(I)} \log^{\frac{1}{\alpha}}\left(1 + \left|t^{(1)} - t^{(2)}\right|_1^{-1}\right)} = +\infty. \tag{5.36}$$

From Theorem 7, we obtain the following pointwise modulus of continuity

Corollary 10 (Pointwise modulus of continuity)

Let τ be an arbitrarily fixed point of $\mathbb{R}^N$.

We assume that the continuous Hurst function H satisfies the **pointwise Hölder condition at τ :** *there exists a finite constant c (which may depend on τ) such that for all t in a neighborhood of τ, one has*

$$\big|H(t)-H(\tau)\big| \le c\big|t-\tau\big|_1^{H(\tau)} \log^{\frac{1}{\alpha}}\big(1+\big|t-\tau\big|_1^{-1}\big). \qquad (5.37)$$

Then, on the event Ω_α^* of probability 1, for all $\eta>0$ and $\varrho>0$, one has

$$\sup_{|t-\tau|_1\le\varrho} \frac{\big|Z(t)-Z(\tau)\big|}{\big|t-\tau\big|_1^{H(\tau)} \log^{\frac{1}{\alpha}+\eta}\big(1+\big|t-\tau\big|_1^{-1}\big)} < +\infty, \qquad (5.38)$$

Theorem 11 (Optimality for pointwise modulus)

Let τ be an arbitrarily fixed point of $\mathbb{R}^N$.

We assume that the continuous Hurst function H satisfies the same **pointwise Hölder condition at τ** *as before.*

Then there exists an event $\widetilde{\Omega}_{\alpha,\tau} \subset \Omega^*_\alpha$ (depending on τ) of probability 1 such that on $\widetilde{\Omega}_{\alpha,\tau}$, one has

$$\limsup_{t \longrightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t-\tau|_1^{H(\tau)} \log^{\frac{1}{\alpha}} \left(1 + |t-\tau|_1^{-1}\right)} = +\infty. \tag{5.39}$$

Remark that, having obtained a global and pointwise optimal moduli of continuity, **we have achieved Goal 1 of the talk.**

In broad terms, a sketch of the proof of Theorem 11

First step :
Denoting by $\mathbb{Z}_e$ the set of even integers, for any given integer $m \geq 2$, and any distinct integers $j_1, j_2, \ldots, j_m$ belonging to $\mathbb{Z}_e$, the m sequences of random variables $\{\varepsilon_{\delta,j_1,k}\}_{(\delta,k)\in\Upsilon^*\times\mathbb{Z}^N}$, $\{\varepsilon_{\delta,j_2,k}\}_{(\delta,k)\in\Upsilon^*\times\mathbb{Z}^N}, \ldots,$ $\{\varepsilon_{\delta,j_m,k}\}_{(\delta,k)\in\Upsilon^*\times\mathbb{Z}^N}$ are independent.
Then, because the $S\alpha S$ distributions of every $\epsilon_{\delta,j,k}^{(\alpha)}$ are heavy-tailed, we use Borel-Cantelli Lemma to obtain the following proposition.

Proposition 5.1

Let $(k_j)_{j\in\mathbb{N}}$ be an arbitrary sequence of elements of $\mathbb{Z}^N$. One has almost surely, for all $\delta \in \Upsilon_*$,

$$\limsup_{j\to+\infty} \frac{|\varepsilon_{\delta,j,k_j}^{(\alpha)}|}{(1+j)^{\frac{1}{\alpha}}} = +\infty. \tag{5.40}$$

Second step : We prove the following proposition

Proposition 5.2

For any fixed $(\widetilde{u},\widetilde{v}) \in \mathbb{R}^N \times (0,1)$*, there is a positive finite deterministic constant* $c(\widetilde{v})$*, depending uniquely on* $\widetilde{v}$*, such that on the event* Ω_α^* *of probability* 1*, one has, for all* $\delta \in \Upsilon_*$*,*

$$\limsup_{j\to+\infty} \frac{\big|\varepsilon^{(\alpha)}_{\delta,j,\lfloor 2^j\widetilde{u}\rfloor}\big|}{(1+j)^{\frac{1}{\alpha}}} \le c(\widetilde{v}) \limsup_{u\to\widetilde{u}} \frac{\big|X(u,\widetilde{v})-X(\widetilde{u},\widetilde{v})\big|}{|u-\widetilde{u}|_1^{\widetilde{v}} \log^{\frac{1}{\alpha}}\big(1+|u-\widetilde{u}|_1^{-1}\big)}, \tag{5.41}$$

where

$$\lfloor 2^j\widetilde{u}\rfloor := \big(\lfloor 2^j\widetilde{u}_1\rfloor,\cdots,\lfloor 2^j\widetilde{u}_N\rfloor\big), \tag{5.42}$$

Therefore, combining both last propositions, for any fixed $(\widetilde{u},\widetilde{v}) \in \mathbb{R}^N \times (0,1)$, there exists an event $\widetilde{\Omega}_{\alpha,\widetilde{u}} \subset \Omega_\alpha^*$ of probability 1, which depends on α and $\widetilde{u}$ but not on $\widetilde{v}$, such that, on $\widetilde{\Omega}_{\alpha,\widetilde{u}}$, one has

$$\limsup_{u\to\widetilde{u}} \frac{\big|X(u,\widetilde{v})-X(\widetilde{u},\widetilde{v})\big|}{|u-\widetilde{u}|_1^{\widetilde{v}} \log^{\frac{1}{\alpha}}\big(1+|u-\widetilde{u}|_1^{-1}\big)} = +\infty. \tag{5.43}$$

Third step : Finally, fixing τ satisfying the **pointwise Hölder condition** for H, one writes the inequality

$$\limsup_{t\to\tau}\frac{\big|Z(t)-Z(\tau)\big|}{|t-\tau|_1^{H(\tau)}\log^{\frac{1}{\alpha}}\left(1+|t-\tau|_1^{-1}\right)}\geq \tag{5.44}$$

$$\limsup_{t\to\tau}\frac{\big|X(t,H(\tau))-X(\tau,H(\tau))\big|}{|t-\tau|_1^{H(\tau)}\log^{\frac{1}{\alpha}}\left(1+|t-\tau|_1^{-1}\right)}-\limsup_{t\to\tau}\frac{\big|X(t,H(t))-X(t,H(\tau))\big|}{|t-\tau|_1^{H(\tau)}\log^{\frac{1}{\alpha}}\left(1+|t-\tau|_1^{-1}\right)}.$$

One gets the conclusion, that means on the event $\widetilde{\Omega}_{\alpha,\tau}$ the red left-hand member is infinite because :

- The term in green is infinite on the event $\widetilde{\Omega}_{\alpha,\tau}$ by second step.
- The term in blue is finite on Ω_α^* combining the Lipschitz-continuity of X with respect to the second variable, and the **pointwise Hölder condition at τ** for H. $\square$

Under a bit stronger assumption on H than the previous local pointwise Hölder condition, it can be shown that the pointwise Hölder modulus of continuity is quasi optimal on a universal event of probability 1 not depending on the location :

Theorem 12 (Quasi optimality on a universal event of probability 1)

There exists a universal event $\widehat{\Omega}_\alpha$ of probability 1 *such that for all* $\tau \in \mathbb{R}^N$ *satisfying*

$$\lim_{t \longrightarrow \tau} \frac{|H(t) - H(\tau)|}{|t-\tau|_1^{H(\tau)}} = 0, \tag{5.45}$$

there exists $\widehat{c} > 0$ *(depending on the function* H*) such that, on* $\widehat{\Omega}_\alpha$*, one has*

$$\limsup_{t \longrightarrow \tau} \frac{|Z(t) - Z(\tau)|}{|t-\tau|_1^{H(\tau)}} \geq \widehat{c} > 0. \tag{5.46}$$

Notice that, under the assumption that the Hurst function H is a locally Lipschitz function on $\mathbb{R}^N$, the conclusion of Theorem 12 is a strictly stronger result than the equality mentioned in Goal 2 and recalled here.

$$\mathbb{P}\big(\forall \tau \in \mathbb{R}^N, \rho_Z(\tau) = H(\tau)\big) = 1.$$

Indeed, under the latter assumption, or more generally when H is a locally Hölder function on $\mathbb{R}^N$ of any arbitrary order $\gamma \in (\overline{H}, 1)$, then the condition (5.45) is satisfied by all point $\tau \in \mathbb{R}^N$, thus (3.7) results from Theorem 12.
Therefore, **we have reached Goal 2.**

Theorem 13 (Estimation of the behavior at infinity)

On the event Ω_α^* *of probability* 1, *for all* $\eta, \varrho > 0$, *one has*

$$\sup_{|t|_1 \geq \varrho} \frac{|Z(t)|}{|t|_1^{H(t)} \log^{\frac{1}{\alpha}+\eta}(1+|t|_1)} < +\infty. \tag{5.47}$$

Moreover, when for some finite constants $H_\infty \in [\underline{H}, \overline{H}] \subset (0,1)$ *and* $c > 0$ *the following inequality holds : for all* $t \in \mathbb{R}^N$,

$$\big|H(t) - H_\infty\big| \leq c\Big(\log(3+|t|_1)\Big)^{-1}, \tag{5.48}$$

then (5.47) can equivalently be reformulated as : on Ω_α^*, *for all* $\eta, \varrho > 0$ *one has*

$$\sup_{|t|_1 \geq \varrho} \frac{|Z(t)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}+\eta}(1+|t|_1)} < +\infty. \tag{5.49}$$

And finally, one results offering optimality for the estimation of behavior at infinity of Z.

Theorem 14 (Optimality of the estimation of the behavior at infinity)

Assume that there exists three finite constants $H_\infty \in [\underline{H}, \overline{H}]$, $\eta_\infty > 0$, and $c > 0$ such that for all $t \in \mathbb{R}^N$, one has

$$\big|H(t) - H_\infty\big| \leq \; c\Big(\log(3 + |t|_1)\Big)^{-1-\eta_\infty}. \tag{5.50}$$

Then there exists an event $\breve{\Omega}_\alpha$ of probability 1, such that on $\breve{\Omega}_\alpha$, one has

$$\limsup_{|t|_1 \to +\infty} \frac{|Z(t)|}{|t|_1^{H_\infty} \log^{\frac{1}{\alpha}}(1 + |t|_1)} = +\infty. \tag{5.51}$$

Theorems 13 and 14 allow us to reach Goal 3 of our talk.

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